ODD GRACEFULL LABELING FOR THE SUBDIVISON OF DOUBLE TRIANGLES GRAPHS

E. M. Badr[1] and M. E. Abdel-aal[2]

[1] Scientific Computing Department, Faculty of Computer Science and Informatics, Benha University, Benha, Egypt.
badrgraph@gmail.com

[2] Mathematics Department, Faculty of Science, Benha University, Benha, Egypt,
mohamed_el77@yahoo.com

ABSTRACT

The aim of this paper is to present some odd graceful graphs. In particular we show that an odd graceful labeling of the all subdivision of double triangular snakes ($2\Delta_k$ -snake). We also prove that the all subdivision of $2m\Delta_1$-snake are odd graceful. Finally, we generalize the above two results (the all subdivision of $2m\Delta_k$ -snake are odd graceful).

KEYWORDS

Graph Labeling; Odd Graceful; Subdivision; Triangular Snakes

1.INTRODUCTION

The graphs considered here will be finite, undirected and simple. The symbols $V(G)$ and $E(G)$ will denote the vertex set and edge set of a graph G respectively. p and q denote the number of vertices and edges of G respectively.

A graph G of size q is odd-graceful, if there is an injection ϕ from $V(G)$ to {0, 1, 2, …, $2q$-1} such that, when each edge xy is assigned the label or weight $|\phi(x) - \phi(y)|$, the resulting edge labels are {1, 3, 5, …, $2q$-1}. This definition was introduced in 1991 by Gnanajothi [1] who proved that the class of odd graceful graphs lies between the class of graphs with α-labelings and the class of bipartite graphs.

Rosa [2] defined a triangular snake(or Δ-*snake*) as a connected graph in which all blocks are triangles and the block-cut-point graph is a path. Let Δ_k -*snake* be a Δ-*snake* with k

blocks while $n\Delta_k$ *-snake* be a Δ *-snake* with k blocks and every block has n number of triangles with one common edge.

Illustration1:

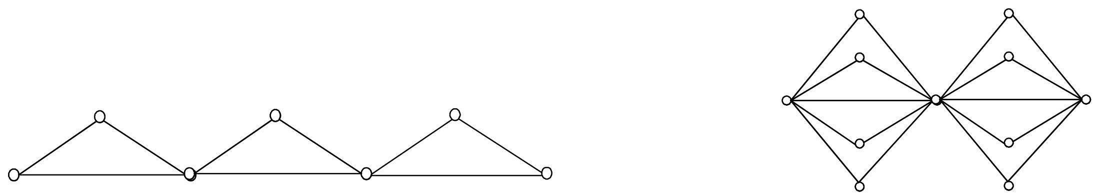

Figure 1: Δ_3-snake $4\Delta_2$-snake

Rosa[2] conjectured that Δ_k *-snake* (a snake with k blocks) is graceful for $n \equiv 0$ or 1 (mod 4) and is nearly graceful otherwise. In 1989 Moulton [3] has proved Rosa's conjecture but using instead of nearly graceful labeling an stronger labeling named almost graceful.

A double triangular snake is a graph that formed by two triangular snakes have a common path. The harmonious labeling of double triangle snake introduced by Xi Yue et al [4]. It is known that the graphs which contain odd cycles are not odd graceful so Badr [5] used the subdivision notation for odd cycle in order to prove that the subdivision of linear triangular snakes are odd graceful. Badr et al [6] proved that the subdivision of ladders $S(L_n)$ is odd graceful.

In this paper we prove that an odd graceful labeling of the all subdivision of double triangular snakes ($2\Delta_k$ *-snake*). We also prove that the all subdivision of $2m\Delta_1$*-snake* are odd graceful. Finally, we generalize the above two results (the all subdivision of $2m\Delta_k$ *-snake* are odd graceful).

2. MAIN RESULTS

***Theorem* 2.1** All the subdivision of double triangular snakes ($2\Delta_k$ *-snake*) are odd graceful.

Proof: Let G = $2\Delta_k$ *-snake* has q edges and p vertices. The graph G consists of the vertices (u_1, $u_2,\ldots,u_{k+1}$), ($v_1,v_2,\ldots,v_k$), ($w_1,w_2,\ldots,w_k$) therefore we get the subdivision of double triangular snakes $S(G)$ by subdividing every edge of double triangular snakes $2\Delta_k$ *-snake* exactly once. Let y_i be the newly added vertex between u_i and u_{i+1} while w_{i1} and w_{i2} are newly added vertices between $w_i\,u_i$ and $w_i\,u_{i+1}$ respectively, where $1 \le i \le k$. Finally,v_{i1} and v_{i2} are newly added vertices between $v_i\,u_i$ and $v_i\,u_{i+1}$respectivley,such that $1 \le i \le k$.
The graph $S(G)$ consists of the vertices (u_1, $u_2,\ldots,u_{k+1}$), ($v_1,v_2,\ldots,v_k$), ($w_1,w_2,\ldots,w_k$), ($w_{11},w_{12},w_{21},w_{22},\ldots,w_{k1},\ldots,w_{k2}$),($v_{11},v_{12},v_{21},v_{22},\ldots,v_{k1},v_{k2}$) and ($y_1,y_2,\ldots,y_k$) as shown in Figure 2. Clearly $S(G)$ has $q = 10k$ edges and $p = 8k + 1$ vertices.

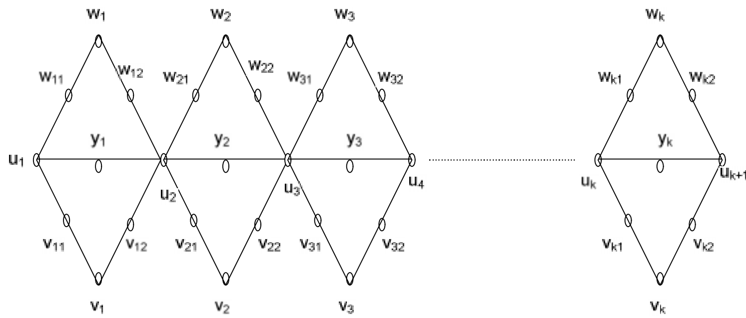

Figure 2: the subdivision of double triangular snakes ($2\Delta_k$ *-snake*)

we prove that the subdivision of double triangular snakes $S(G)$ is odd graceful.
Let us consider the following numbering ϕ of the vertices of the graph G :

$\phi(u_i) = 6(i - 1)$ $\quad 1 \le i \le k+1$
$\phi(y_i) = 2q - 14i + 11$ $\quad 1 \le i \le k$
$\phi(v_i) = 6i - 4$ $\quad 1 \le i \le k$
$\phi(v_{ij}) = 2q - 14i - 8j + 17$ $\quad 1 \le i \le k$, $j = 1, 2$
$\phi(w_i) = 6i + 4$ $\quad 1 \le i \le k$
$\phi(w_{ij}) = 2q - 14i - 6j + 19$ $\quad 1 \le i \le k$, $j = 1, 2$

(a) $$\underset{v \in V}{Max}\ \phi(v) = \max\left\{ \max_{1 \le i \le k+1} 6(\text{ i - 1 }), \max_{1 \le i \le k} \text{ 2q - 14i + 11 - 1}, \max_{1 \le i \le k} \text{ 6i - 4}, \max_{\substack{1 \le j \le 2 \\ 1 \le i \le k}} \text{ 2q - 14i - 8j + 17}, \max_{1 \le i \le k} \text{ 6i + 4}, \max_{\substack{1 \le j \le 2 \\ 1 \le i \le k}} \text{ 2q- 14i -6j + 19} \right\}$$

= $2q - 1$, the maximum value of all odds .

Thus $\phi(v) \in \{0, 1, 2, \ldots, 2q - 1\}$.

(b) Clearly ϕ is a one – to – one mapping from the vertex set of G to $\{0, 1, 2, \ldots, 2q-1\}$.

(c) It remains to show that the labels of the edges of G are all the odd integers of the interval $[1, 2q-1]$.

The range of $|\phi(u_i) - \phi(w_{i1})| = \{2q - 20i + 19;\ i = 1,2,\ldots,k\} = \{2q-1, 2q-21, \ldots\ldots, 19\}$
The range of $|\phi(u_i) - \phi(y_i)| = \{2q - 20i + 17;\ i = 1,2,\ldots.k\} = \{2q-3, 2q-23, \ldots\ldots, 17\}$
The range of $|\phi(u_i) - \phi(v_{i1})| = \{2q-20i+15;\ i = 1,2,3,\ldots., k\} = \{2q-5, 2q-25, \ldots\ldots, 15\}$
The range of $|\phi(v_{i1}) - \phi(v_1)| = \{2q-20i+13;\ i = 1,2,3,.., k\} = \{2q-7, \ldots, 13\}$
The range of $|\phi(y_i) - \phi(u_{i+1})| = \{2q-20i+11;\ i = 1,2,3,\ldots, k\} = \{2q-9, \ldots\ldots, 11\}$

The range of $|\phi(w_{i1})-\phi(w_i)| = \{2q - 20i + 9 ; i = 1,2,\dots, k\} = \{2q - 11,\dots,9\}$
The range of $|\phi(w_{i2})-\phi(u_{i+1})| = \{2q - 20i + 7 ; \mathrm{i} = 1,2,\dots, k\} = \{2q - 13,\dots,7\}$
The range of $|\phi(v_i)-\phi(v_{i2})| = \{2q - 20i + 5 ; i = 1,2,3,\dots, k\} = \{2q - 15,\dots,5\}$
The range of $|\phi(w_{i2})-\phi(w_i)| = \{2q - 20i + 3 ; i = 1,2,3,\dots, k\} = \{2q - 17,\dots,3\}$
The range of $|\phi(v_{i2})-\phi(u_{i+1})| = \{2q - 20i + 1 ; i = 1,2,3,\dots, k\} = \{2q - 19,\dots,1\}$
Hence $\{|\phi(\mathrm{u}) - \phi(\mathrm{v})| : uv \in E\} = \{1, 3, 5,\dots, 2q-1\}$ so that the subdivision of double triangular snakes ($2\Delta_k$ *-snake*) are odd graceful. ■

Illustration2:

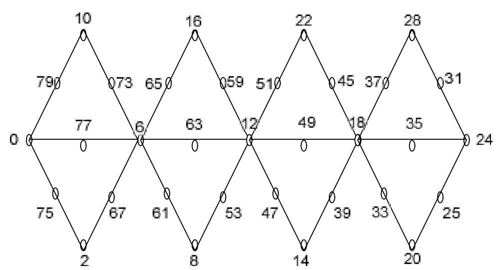

Figure 3: odd-graceful labeling of the graph $S(2\Delta_4$ *-snake*).

***Theorem* 2.2** All the subdivision of $2m\Delta_1$*-snake* are odd-graceful, where $m \geq 1$.

Proof :

Let $G = 2m\Delta_1$*-snake* has q edges and p vertices. The graph G consists of the vertices (u_1, u_2), ($v_1^1, v_1^2, \dots, v_1^m$), ($w_1^1, w_1^2, \dots, w_1^m$) therefore we get the subdivision of the graph G, $S(G)$, by subdividing every edge of the graph $G = 2m\Delta_k$ *-snake* exactly once. Let y_1 be the newly added vertex between u_1 and u_2 while w_{11}^i and w_{12}^i are newly added vertices between $w_1^i u_1$ and $w_1^i u_2$ respectively. Finally , v_{11}^i and v_{12}^i are newly added vertices between $v_1^i u_1$ and $v_1^i u_2$ respectively where $i = 1, 2, \dots, m$.
The graph $S(G)$ consists of the vertices (u_1, u_2), ($v_1^1, v_1^2, \dots, v_1^m$), ($w_1^1, w_1^2, \dots, w_1^m$), y_1 , (w_{11}^i, $w_{12}^i, \dots, w_{12}^m$) and ($v_{11}^i, v_{12}^i, \dots, v_{12}^m$) as shown in Figure 4. Clearly $S(G)$ has $q = 8m + 2$ edges and $p = 6m + 3$ vertices.

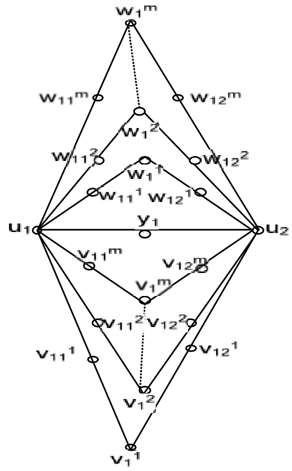

Figure 4: the subdivision of $2m\Delta_1$-*snake*

Let us consider the following numbering ϕ of the vertices of the graph G :

$\phi(u_i) = (4m + 2)(i - 1)$ $\quad i = 1,2$

$\phi(y_1) = 14m + 3$

$\phi(v_i^l) = 4l - 2$ $\quad 1 \le l \le m$

$\phi(w_1^l) = 4(l + m) + 2$ $\quad 1 \le l \le m$

$\phi(v_{1j}^l) = 20m + 2l + 3 - (8m + 2)j$ $\quad 2 \le l \le m, \; j = 1,2$

$\phi(v_{1j}^1) = (18m + 5) - (6m + 2)j$ $\quad j = 1,2$

$\phi(w_{1j}^l) = (18m + 2l + 5) - (4m + 2)j$ $\quad j = 1,2 \;, \; 2 \le l \le m$

The edge labels well be as follows:

The vertices u_1 and w_{11}^l, $1 \le l \le m$, induce the edge labels = $\{14m + 2l + 3 , 1 \le l \le m\}$ = $\{14m + 5, 14m + 7, \ldots, 16m + 3\}$

The vertices u_1 and y_1 induce the edge labels $\{14m + 3\}$

The vertices u_1 and v_{11}^l; $1 \le l \le m$, induce the edge labels $\{12m + 3, 12m + 2l + 1 ; 2 \le l \le m\}$ $=\{12m + 3, 12m + 5, \ldots, 14m + 1\}$

The vertices v_{11}^l and v_1^l ,y_1 and u_2 , w_{11}^l and w_1^l induce the edge labels ; $1 \le l \le m$: $\{12m + 3 - 2l: 1 \le l \le m\}, \{10m + 1\}, \{10m - 2l + 1; 1 \le l \le m\} = \{12m + 1, 12m - 1, \ldots, 10m + 1, 10m - 1, \ldots, 8m + 1\}$.

The vertices w_{12}^l and u_2 induce the edge label$\{6m + 2l - 1\}; 1 \le l \le m\} = \{6m + 1, 6m + 3, \ldots, 8m - 1\}$

The vertices v_1^1 and v_{12}^1 induce the edge label $\{6m - 1\}$.

The vertices w_{12}^l and w_i^l, induce the edge labels $\{6m - 2l - 1; \; 1 \le l \le m\} = \{6m - 3, 6m - 5, \ldots, 4m - 1\}$.

The vertices v_{12}^l and v_1^l ; $2 \le l \le m$ induce the edge labels = $\{4m - 2l + 1; 2 \le l \le m\}$

$= \{4m - 3, 4m - 5, \ldots, 2m + 1\}$.

The vertices v_{12}^{1} and u_2 induce the edge label $\{2m - 1\}$.

Finally the vertices v_{12}^{l} and u_2 induce the edge labels$\{2l\text{-}3\ ;\ 2 \le l \le m\} = \{1,3,5,\ldots,2m\text{-}3\}$.

Hence the graph $S(2m\Delta_1\text{-}snake)$ is odd-graceful for each $m \geq 1$. ■

***Theorem* 2.3** All subdivision of $2m\Delta_k$ *-snake* are odd-graceful

Proof.

Let G=$2m\Delta_k$ *-snake* has q edges and p vertices. The graph G consists of the vertices (u_1, $u_2,\ldots,u_{k+1}$), ($v_1^1, v_1^2, \ldots, v_1^m$), ($v_2^1, v_2^2, \ldots, v_2^m$), . . ., ($v_k^1, v_k^2, \ldots, v_k^m$), ($w_1^1, w_1^2 \ldots, w_1^m$), ($w_2^1, w_2^2 \ldots, w_2^m$), . . ., ($w_k^1, w_k^2 \ldots, w_k^m$) therefore we get the subdivision of double triangular snakes $S(G)$ by subdividing every edge of $2m\Delta_k$ *-snake* exactly once. Let y_1 be the newly added vertex between u_1 and u_2 while w_{i1}^{j} and w_{i2}^{j} are newly added vertices between $w_i^j\,u_i$ and $w_i^j\,u_{i+1}$ respectively. Finally , v_{i1}^{j} and v_{i2}^{j} are newly added vertices between $v_i^j\,u_i$ and $v_i^j\,u_{i+1}$ - respectively where $i = 1, 2, \ldots, k$ *and* $j = 1, 2, 3, \ldots, m$ (Figure 5). Clearly $S(G)$ has $q = k\,(8m + 2)$ edges.

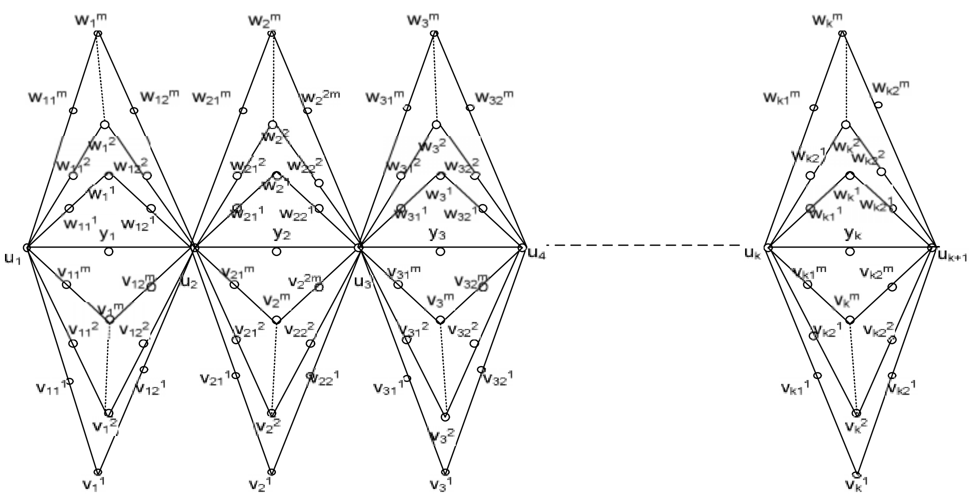

Figure 5: the subdivision of $2m\Delta_k$ *-snake*

Let us consider the following numbering ϕ of the vertices of the graph G :

$\phi\ (u_i) = (4m + 2)(\ i - 1)$ $\quad 1 \le i \le k$

$\phi\ (w_i^l) = (4m+2)i + 4l$ $\quad 1 \le i \le k\ ,\ 1 \le l \le m$

$\phi\ (v_i^l) = (4m+2)i + 4(l\text{-}m\text{-}1)$ $\quad 1 \le i \le k\ ,\ 1 \le l \le m$

$\phi\ (\ y_i\ ) = 2q\text{-}(12m+2)i + 10m+1\ ;\ 1 \le i \le k\ ,\ 1 \le j \le 2$

$\phi\ (w_{ij}^{l})\ = 2q\text{-}(12m+2)i - (4m+2)j + 14m + 2l + 3;\ \ 1 \leq i \leq k\ ,\ 1 \leq j \leq 2,\ 1 \leq l \leq m$

$\phi\ (v_{ij}^{l})\ = 2q\text{-}(12m+2)i - (8m+2)j + 16m + 2l + 1;\ \ 1 \leq j \leq 2,\ 1 \leq l \leq m$

$\phi\ (v_{ij}^{1})\ = 2q\text{-}(12m+2)i - (6m+2) + 14m + 3\ ;\ 1 \leq i \leq k\ ,\ 1 \leq j \leq 2$

In a view of the above defined labeling pattern ϕ is odd-graceful for the graph $S(G)$. hence $S(G)$ is odd-graceful for all $m \geq 1$, $k \geq 1$.

Illustration 2,3 :

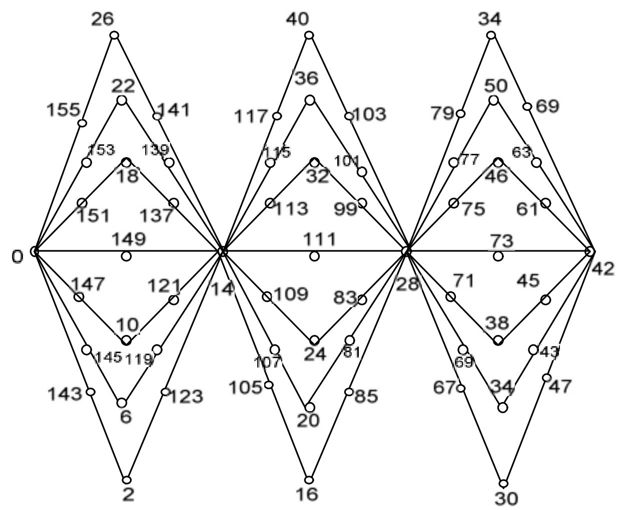

Figure 6: odd-graceful labeling of the graph $6\Delta_3$ *-snake*

3. CONCLUSION

Graceful and odd graceful of a graph are two entirely different concepts. A graph may posses one or both of these or neither. In the present work we show that an odd graceful labeling of the all subdivision of double triangular snakes ($2\Delta_k$ *-snake*). We also proved that the all subdivision of $2m\Delta_1$*-snake* are odd graceful. Finally, we generalized the above two results (the all subdivision of $2m\Delta_k$ *-snake* are odd graceful).

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Authors

Sayed Badr received the B.Sc. (Mathematics) the M.Sc. (Pure Mathematics) degree from Benha University, Benha, Egypt in 1994, 2000 respectively. Also, he received Ph.D. (Computer Science) degree from Department of Applied Informatics, Macedonia University, Thessaloniki, Greece, in 2006. He is a University lecturer of Computer Science with the Benha University, Faculty of Computer Science, Department of Scientific Computing. His current research is Parallel Programming, linear programming and Graph Theory.

Mohamed Elsayed Abdel-Aal received the B.Sc. (Mathematics) the M.Sc. (Pure Mathematics- Abstract Algebra) degree from Benha University, Benha, Egypt in 1999, 2005 respectively. Also, he received Ph.D. (Pure Mathematics) degree from Faculty of Mathematics, Tajik National University, Tajikistan, in 2011. He is a University lecturer of Pure Mathematics with the Benha University, Faculty of Science, Department of Pure Mathematics. His current research is Ordinary – partial differential equations Graph Theory and Abstract Algebra